CONTINUITY AND DIFFERENTIABILITY

(B.Sc.-II, Paper-III)

Group- A

(Real Analysis)

Topic: - *Continuity of a function of one variable,*
Types of Discontinuity,
Properties of Continuous function,
Differentiability,

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Continuity :→

Definition :→ (Continuous Function at a point)

Suppose $f : D(f) \longrightarrow \mathbb{R}$, and $x_0 \in D(f)$.

Then f is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

Remark :- ① The function f is said to be continuous on interval I if f is continuous at every $a \in I$.

(ii) f is continuous at $a \in I$ if and only if for every $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$|f(x) - f(a)| < \epsilon$ whenever $x \in I$, $|x - a| < \delta$.

THEOREM (Sequential Criterion for Continuity of f at x_0)

Let $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. Then f is continuous at $x_0 \in D$ if and only if every sequence (x_n) in D s.t. $\lim_{n \to \infty} x_n = x_0$ we have $\lim_{n \to \infty} f(x_n) = f(x_0)$

Proof :→ ($\Rightarrow$) Let us first suppose that f is continuous at x_0 and let (x_n) be a sequence in D s.t. $\lim_{n \to \infty} x_n = x_0$

∴ For given $\epsilon > 0$.

∵ f is continuous at x_0, $\exists\ \delta > 0$ s.t.

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \quad \text{—— ①}$$

∵ $\lim_{n \to \infty} x_n = x_0$, Therefor $\exists$ a positive integer

N s.t.

$$|x_n - x_0| < \delta, \ \forall\, n \geq N. \quad — (2)$$

putting $x = x_n$ in (1), we have

$$|x_n - x_0| < \delta \Rightarrow |f(x_n) - f(x_0)| < \epsilon \quad — (3)$$

From (2) & (3), we have

$$|f(x_n) - f(x_0)| < \epsilon, \ \forall\, n \geq N.$$

i.e; $\lim_{n\to\infty} f(x_n) = f(x_0)$.

<u>proved.</u>

($\Leftarrow$) Suppose that for every sequence (x_n) in D s.t. $\lim_{n\to\infty} x_n = x_0$ then $\lim_{n\to\infty} f(x_n) = f(x_0)$

Let us now suppose that f is not continuous at x_0.

We shall show that $\exists$ a sequence (x_n) such that $\lim_{n\to\infty} x_n = x_0$, but $\lim_{n\to\infty} f(x_n) \neq f(c)$.

$\because$ f is not continuous at x_0, then $\exists$ an $\epsilon > 0$ s.t. for every $\delta > 0$, $\exists$ an $x \in D$ s.t.

$$|x - x_0| < \delta \quad \& \quad |f(x) - f(x_0)| \geq \epsilon.$$

By taking $\delta = \frac{1}{n}$, we find for ~~each~~ each positive integer n, $\exists$ an $x_n \in D$ s.t.

$$|x_n - x_0| < \frac{1}{n} \text{ and } |f(x_n) - f(c)| \geq \epsilon$$

Then $\lim_{n\to\infty} x_n = x_0$ but $\lim_{n\to\infty} f(x_n) \neq f(x_0)$

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Example:→ A function $f(x)$ is defined as follows:

$$f(x) = x \sin\frac{1}{x} \quad \text{for } x \neq 0$$

$$= 0 \quad \text{for } x = 0$$

show that $f(x)$ is continuous at $x=0$.

Proof:→ Let $\epsilon > 0$ be given.

Choose $\delta := \epsilon$. Then if $0 < |x-0| < \delta$
i.e; if $0 < |x| < \delta$.

We have

$$|f(x) - 0| = |x \sin\tfrac{1}{x} - 0|$$

$$= |x| \, |\sin\tfrac{1}{x}|$$

$$\leq |x| \quad \left(\because \left|\sin\tfrac{1}{x}\right| \leq 1\right)$$

$$< \delta := \epsilon$$

Hence $\lim\limits_{x\to 0} f(x) = 0 = f(0)$

Therefor, $f(x)$ is continuous at $x=0$.

Example②:- A function $f(x) = x^n$ defined over $\mathbb{R}$ is continuous for every $c \in \mathbb{R}$ (where n is a positive integer) for

$$\lim_{x\to c} f(x) = \lim_{x\to c} x^n = c^n = f(c)$$

$\therefore f(x)$ is continuous $\forall\, c \in \mathbb{R}$.

proved.

Example ③ The function $f(x) = ax^n$ (where a is in $\mathbb{R}$) is continuous for every c in $\mathbb{R}$. (where n is positive integer) for

$$\lim_{x\to c} f(x) = \lim_{x\to c} ax^n = ac^n = f(c)$$

Example ④ Every polynomial over $\mathbb{R}$ is continuous at every point c in $\mathbb{R}$.

Solution :→ Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial (where $a_0, \ldots, a_n \in \mathbb{R}$) over $\mathbb{R}$.

Then

$$\lim_{x\to c} p(x) = \lim_{x\to c} (a_n x^n + \cdots + a_1 x + a_0)$$

$$= a_n c^n + \cdots + a_1 c + a_0$$

$$= p(c)$$

Continuity: →

Example① Prove that the Dirichlet's function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational.} \end{cases}$$

is discontinuous at every point.

Solution: →

First, Let c be a rational number.

For each positive integer n, Let x_n be an irrational number s.t. $|x_n - c| < \frac{1}{n}$, then $(x_n) \to c$.

But $f(x_n) = -1$ for all n, so that

$\lim_{n\to\infty} f(x_n) = -1 \neq f(c) = 1$.

Hence f is discontinuous at c.

Next, let i be an irrational number.

For each positive integer n, Let us choose a rational number y_n s.t. $|y_n - i| < \frac{1}{n}$ Then $(y_n) \to i$

But $f(y_n) = 1 \ \forall n$

So, that $\lim_{n\to\infty} f(y_n) = 1 \neq f(i) = -1$

Hence f is discontinuous at i.

Hence f is discontinuous at every point $x \in \mathbb{R}$.

Example: → Let f be defined on $[-1,1]$ by setting

$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is continuous only at $x=0$.

Solution: → Let c be any point of $[-1,1]$.

For each positive integer n, choose a rational number x_n and an irrational number y_n, both in $[-1,1]$ s.t.

$$|x_n - c| < \frac{1}{n}, \quad |y_n - c| < \frac{1}{n}$$

Then

$$\lim_{n\to\infty} x_n = c = \lim_{n\to\infty} y_n$$

If f is continuous at c, we must have

$$\lim_{n\to\infty} f(x_n) = f(c) = \lim_{n\to\infty} f(y_n)$$

Now, $f(x_n) = 0 \ \forall n$

and $f(y_n) = y_n \ \forall n$.

Hence we must have

$$0 = f(c) = \lim_{n\to\infty} y_n$$

i,e; $0 = f(c) = c$

$$\Rightarrow c = 0$$

Thus 0 is the only possible point of continuity. We shall now show that f is actually continuous at 0.

Given $\epsilon > 0$, take $\delta = \frac{\epsilon}{2}$. Then

$|x| < \delta \Rightarrow |f(x) - f(0)| = 0$, if x is rational.

and $|x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \delta < \epsilon$, if x is irrational.

Thus $|x| < \delta \Rightarrow |f(x) - f(0)| < \epsilon$

Hence f is continuous at 0.

Example ③ :→ Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ by setting $f(x) = |x|$, $\forall x \in \mathbb{R}$.

Show that f is continuous on $\mathbb{R}$ (i.e, the modulus function on $\mathbb{R}$ is continuous on $\mathbb{R}$).

Proof :→ Let $x \in \mathbb{R}$ be arbitrary and Let (x_n) be any sequence in $\mathbb{R}$ converging to x

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} |x_n| = |x| = f(x)$$

Since, $(x_n) \to x \Rightarrow f(x_n) \to f(x)$

$\therefore$ f is continuous at x.

$\therefore$ f is continuous at each point of $\mathbb{R}$.

Example ④ :→ (Continuity of a function at a point is local but not global property)

Let f be the function defined on $]0,1[$ by setting

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ \& } (p,q) = 0 \end{cases}$$

Then show that f is continuous at each irrational point and discontinuous at each rational point.

Proof:→ Let c be any rational number in $]0,1[$,

Let $c = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ & $(p,q) = 1$.

For each positive integer n, ~~steep~~ select an irrational x_n in $]0,1[$ s.t.

$$|x_n - c| < \frac{1}{n}$$

Then, $(x_n) \to c$ Also, $f(x_n) = 0$ $\forall n$, so that

$$\lim_{n\to\infty} f(x_n) = 0 \neq f(c) \quad \because f(c) = \frac{1}{q}.$$

Thus f is not continuous at c.

Next, Let i be an irrational number in $]0,1[$ and let $\epsilon > 0$ be given choose a positive integer n s.t. $\frac{1}{n} < \epsilon$.

Now there are only finitely many rational numbers $\frac{p}{q}$ in $]0,1[$ having their denominators less than n.

$\therefore$ We can find a $\delta > 0$ s.t. every rational number in $]i-\delta, i+\delta[$ has its denominator $\geq n$.

Then $|x-i| < \delta \Rightarrow |f(x) - f(i)| = |f(x) - 0| = |f(x)| = 0$

if x is irrational.

and $|x-i| < \delta \Rightarrow |f(x) - f(i)| = |f(x) - 0|$

$$= |f(x)| < \frac{1}{n} < \epsilon,$$

if x is rational.

Thus $|x-i| < \delta \Rightarrow |f(x) - f(i)| < \epsilon$

Hence f is continuous at i.

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Continuity

Types of discontinuities :→

→ A function f is said to ~~be~~ discontinuous at a point c of its domain if f is not continuous at c.

→ The discontinuity of f at c can arise in either of the following two ways,

(i) $\lim_{x \to c} f(x)$ exists and different from $f(c)$

(ii) $\lim_{x \to c} f(x)$ does not exists.

(I) A function f is said to have a removable discontinuity at a point c of its domain if $\lim_{x \to c} f(x)$ exists but

It is not equal to $f(c)$.

(II) (i) f is said to have a discontinuity of the first kind from the left at c

if $\lim_{x \to c-0} f(x)$ exists but not equal to $f(c)$

(ii) f is said to have a discontinuity of the first kind from the right if

$\lim_{x \to c+0} f(x)$ exists but is not equal to $f(c)$.

Remark :→ f is said to have a discontinuity of the first kind at c (or ordinary discontinuity at c) if $\lim_{x \to c-0} f(x)$ and $\lim_{x \to c+0} f(x)$ exists but are unequal.

(III) (i) f is said to have a discontinuity of the seccond kind from the left at c if $\lim_{x \to c-0} f(x)$ does not exists.

(ii) f is said to have a discontinuity of the second kind from the right at c if $\lim_{x \to c+0} f(x)$ does not exists.

Remark :→ f is said to have a discontinuity of the second kind at c if neither of $\lim_{x \to c-0} f(x)$ & $\lim_{x \to c+0} f(x)$ exists.

Example ① Let f be a function defined on $\mathbb{R}$ by setting

$$f(x) = \frac{\sin x}{x}, \text{ if } x \neq 0$$

$$f(0) = 0$$

show that f has a removable singularity at $x = 0$.

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Solution :→

Here $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1.$

$\because \lim_{x\to 0} f(x)$ exists but is not equal to $f(0) = 0.$

$\therefore$ f is not continuous at $x = 0$,

In fact f has a removable discontinuity at $x = 0$.

The discontinuity at $x = 0$ may be removed by redefining the function at $x = 0$ s.t. $f(0) = 1$.

Example ②: → The function $f(x) = \frac{1}{2 + e^{\frac{1}{x}}}$ defined on $\mathbb{R}$ has a discontinuity of first kind at $x = 0$ (or ordinary discontinuity at $x = 0$)

Solution: -

Since $\lim_{x\to 0-0} f(x) = \frac{1}{2}$ & $\lim_{x\to 0+0} f(x) = 0$

And thus $\lim_{x\to 0-0} f(x)$ & $\lim_{x\to 0+0} f(x)$ exists but are unequal.

Example ③ A function $f(x)$ is defined as follows:

$f(x) = 1, 0$ or -1 according as $x > 0 =$ or < 0.

Show that it is discontinuous at $x = 0$.

What type of discontinuity at $x = 0$?

Solution: - $\lim_{x\to 0+0} f(x) = \lim_{x\to 0+0} 1 = 1,$

$\lim_{x\to 0-0} f(x) = \lim_{x\to 0-0} (-1) = -1; \quad f(0) = 0$

Hence f has a discontinuity of the from both the sides at $x=0$.

Example ④ Let f be the function defined on $\mathbb{R}$ by setting

$$f(x) = \sin\frac{1}{x}, \quad \text{if } x \neq 0$$
$$= 0, \quad \text{if } x = 0$$

→ Then neither $\lim_{x \to 0-0} f(x)$ nor $\lim_{x \to 0+0} f(x)$ exists.

Hence f has a discontinuity of the second kind at $x=0$.

Continuity :→

Algebraic operations on continuous functions :→

THEOREM :→ Let f, g be continuous from a subset D of $\mathbb{R}$ into $\mathbb{R}$. If f and g are continuous at $c \in D$, then each of the following functions is also continuous at c.

(i) $f+g$ (ii) $f-g$ (iii) fg

(iv) $\frac{f}{g}$, provided that $g(c) \neq 0$ (v) $|f|$

(vi) kf for each $k \in \mathbb{R}$.

Proof :→ $\because$ f and g are continuous at c so

$\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$. Hence.

(i) $\lim_{x \to c} (f+g)(x) = \lim_{x \to c} [f(x) + g(x)]$

$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c)$$
$$= (f+g)(c)$$

Hence $f+g$ is continuous at c.

(ii) $\lim_{x \to c} (f-g)(x) = \lim_{x \to c} [f(x) - g(x)]$

$$= \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$
$$= f(c) - g(c) = (f-g)(c)$$

Hence $f-g$ is continuous at c.

(iii) $\lim_{x \to c} (fg)(x) = \lim_{x \to c} [f(x) \cdot g(x)]$

$$= \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$$
$$= f(c) \cdot g(c) = (f \cdot g)(c)$$

Hence (fg) is continuous at c.

(iv) $\lim_{x \to c} \left(\frac{f}{g}\right) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)} = \frac{f}{g}(c)$

provided $g(c) \neq 0$.

Hence $\frac{f}{g}$ is <u>continuous</u> at c provided $g(c) \neq 0$.

(v) $\lim_{x \to c} |f|(x) = \lim_{x \to c} |f(x)| = \left|\lim_{x \to c} f(x)\right| = |f(c)|$

$= |f|(c)$

Hence $|f|$ is <u>continuous</u> at c.

(vi) For $k \in \mathbb{R}$, $\lim_{x \to c} (kf)(x) = \lim_{x \to c} k \cdot f(x)$

$= k \lim_{x \to c} f(x)$

$= k \cdot f(c)$

$= (kf)(c)$

Hence (kf) is <u>continuous</u> at c.

<u>THEOREM</u>:→ If g is continuous at c & f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

<u>Proof</u>:→ Let $\epsilon > 0$ be given. Since f is continuous at $g(c)$, $\exists\ \delta_1 > 0$ s.t.

$|x - g(c)| < \delta_1$, then

$|f(x) - f(g(c))| < \epsilon$

Again since g is continuous at c, $\exists\ \delta > 0$ s.t. if $|x - c| < \delta$, then $|g(x) - g(c)| < \delta_1$.

Now suppose ~~that~~ $|x - c| < \delta$. Then

$|g(x) - g(c)| < \delta_1,$

so $|f(g(x)) - f(g(c))| < \epsilon$ &

Hence fog is continuous at $\underline{c}$.

Properties of continuous functions

Definition:→ Let f be a real-valued function defined on $[a,b]$. We say that f is continuous on $[a,b]$ if f is continuous at every point of $[a,b]$.

{ at the end-points a & b defined as

$$\lim_{x \to a+0} f(x) = f(a) \quad \& \quad \lim_{x \to b-0} f(x) = f(b) \}$$

Definition:→ A real-valued function defined on a subset D of $\mathbb{R}$ is said to be a bounded on D if $\exists$ a number M s.t.

$$|f(x)| < M, \ \forall x \in D.$$

THEOREM:→ If a function f is continuous on a closed interval $[a,b]$, then f is bounded on $[a,b]$.

Proof:→ Let f be continuous on $[a,b]$.

Suppose f is not bounded on $[a,b]$.

Then to each natural number n ~~[illegible]~~ $\exists$ a point x_n of $[a,b]$ s.t.

$$|f(x_n)| > n \quad \text{———} \quad (1)$$

$\because x_n \in [a,b]$ for each n, $a \leq x_n \leq b$, $\forall n$.

Hence (x_n) is a bounded real sequence.
($\because$ Every bounded real sequence has a convergent subsequence)

$\therefore$ Let (x_{n_k}) be a convergent subsequence of (x_n).

Let $c = \lim\limits_{k\to\infty} x_{n_k}$.

Then $a \leq c \leq b$ i.e. $c \in [a,b]$.

$\because f$ is continuous on $[a,b]$,

$\Rightarrow f$ is continuous at $x = c$.

$\therefore f(c) = \lim\limits_{k\to\infty} f(x_{n_k})$ ——— (2)

But since $f(x_{n_k}) > n_k$ & $n_k \to \infty$ as $k \to \infty$.

$\therefore \lim\limits_{k\to\infty} f(x_{n_k}) = \infty$ ——— (3)

(2) & (3) contradict each other.

Hence f must be bounded on $[a,b]$.

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Example:→ Let f be defined by $f(x) = \frac{1}{x}$; $\forall x \in]0,1]$

f is continuous on $]0,1]$ but not bounded above in $]0,1]$.

continuity

THEOREM: → If f is continuous on $[a,b]$, $\exists$ points c and d in $[a,b]$ s.t.

$$f(c) \leq f(x) \leq f(d) \quad \forall\, x \in [a,b].$$

That is, if f is continuous on $[a,b]$, then f attains its supremum and infimum on $[a,b]$.

Proof: → $\because$ f is continuous on $[a,b]$

$\Rightarrow$ f is bounded on $[a,b]$.

Let $M = \text{l.u.b}\{f(x) : x \in [a,b]\}$.

We must show that $f(d) = M$ for some $d \in [a,b]$.

Suppose this fails, then $f(x) < M$ $\forall\, x \in [a,b]$

Let $g(x) = \dfrac{1}{M - f(x)}$ for $x \in [a,b]$.

Then g is continuous on $[a,b]$.

Hence g is bounded on $[a,b]$.

And thus $\exists$ a positive number K s.t. $g(x) < K$, $\forall\, x \in [a,b]$.

We have $\dfrac{1}{M - f(x)} < K$, $\forall\, x \in [a,b]$

$$\Rightarrow f(x) < M - \frac{1}{K}, \quad \forall\, x \in [a,b]. \quad -(1)$$

$$\because M - \frac{1}{K} < M$$

The inequality (1) contradicts the fact that M is the least upper bound of the set $\{f(x) : x \in [a,b]\}$

$\therefore$ $f(d) = M$ for some $d \in [a,b]$.
Hence f attains maximum at some $d \in [a,b]$.

Next, let $L = g.l.b\{f(x) : x \in [a,b]\}$

We must show that $f(c) = L$ for some $c \in [a,b]$.

Suppose this fails.
Then $f(x) > L$ for every $x \in [a,b]$.

Let $h(x) = \dfrac{1}{f(x) - L}$ for $x \in [a,b]$.

Then h is continuous on $[a,b]$, so h is bounded on $[a,b]$, and thus $\exists$ a positive number D s.t.

$$h(x) < D \text{ for every } x \text{ in } [a,b].$$

$$\Rightarrow \frac{1}{f(x) - L} < D \text{ for } \forall x \in [a,b].$$

$$\Rightarrow f(x) > L + \frac{1}{D}, \forall x \in [a,b]. \quad \text{— (2)}$$

$$\because L + \frac{1}{D} > L$$

Inequality contradicts the fact that L is the greatest lower bound of the set

$$\{f(x) : x \in [a,b]\}$$

Thus $f(c) = L$ for some c in $[a,b]$.
Hence f attains minimum at some point $c \in [a,b]$.

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Remark: $\rightarrow$ If the hypothesis of the above theorem is not valid, the conclusion may fail to hold.

Example ①: $\rightarrow$ Let f be defined by $f(x) = x \; \forall x \in [0,1[$.

f is continuous in $[0,1[$, it is bounded above in $[0,1[$, $l.u.b f = 1$, but there is no point of $[0,1[$, at which $f(x) = 1$.

Observe that $D(f)$ is not closed interval.

Example ② Let f be defined by

$$f(x) = \frac{1}{1+x^2}, \quad \forall x \in \mathbb{R}.$$

f is continuous on $\mathbb{R}$ & also bounded on $\mathbb{R}$, $l.u.b f = 1$ & $g.l.b f = 0$.

While lub f is attained at $x=0$ ($f(0) = 1 = l.u.b f$), glb f is not attained, i.e; there is no point of $\mathbb{R}$ at which $f(x) = glb f = 0$.

Observe that the domain of f is $\mathbb{R}$ which is not closed interval.

THEOREM: $\rightarrow$ If a function f is continuous on a closed interval $[a,b]$ and $f(a)$ & $f(b)$ are of opposite signs then $\exists$ at least one point $c \in [a,b]$ s.t. $f(c) = 0$

Proof: - Try yourself.

THEOREM (Intermediate value theorem)

If a function f is continuous on $[a,b]$ & $f(a) \neq f(b)$, Then f assumes every value between $f(a)$ & $f(b)$.

Proof:→ Let K be any real number between $f(a)$ & $f(b)$.

We shall show that $\exists$ a real number p in $[a,b]$ s.t. $f(P) = K$.

Let g be a function defined on $[a,b]$ by

$$g(x) = f(x) - K$$

clearly g is continuous on $[a,b]$.

Also, $g(a) = f(a) - K$, $g(b) = f(b) - K$

$\because$ K lies between $f(a)$ and $f(b)$, therefor $g(a)$ & $g(b)$ are of opposite signs.

$\therefore$ $\exists$ $p \in [a,b[$ s.t. $g(P) = 0$

Hence $f(P) - K = 0$

$$\Rightarrow f(P) = K$$

Thus f assumes the value K at the point $p \in]a,b[$.

proved.

Differentiability of real functions of a Single variable :→

DEFINITION :→ Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a real-valued function defined at all points of an open interval $]a,b[$ of the real line

For each $x \in]a,b[$, we define.

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

if the limit exists & is finite.

Definition :→ Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a real-valued function defined at all points of the closed interval $[a,b]$ of the real line

$$f'(x+0) = \lim_{h \to 0+0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x-0) = \lim_{h \to 0-0} \frac{f(x+h) - f(x)}{h}$$

When $f'(x+0)$ and $f'(x-0)$ are defined, they are called the right hand & left hand derivatives of f at x respectively.

THEOREM :→ A function which is (finitely) differentiable at a point is necessarily continuous at that point.

Proof :→ Let a function f be finitely differentiable at c, Then

$$f'(c) = \lim_{y \to c} \frac{f(y) - f(c)}{y - c}$$ exists & is finite.

Now, $f(y) - f(c) = \frac{f(y) - f(c)}{y - c} (y-c)$, if $y \neq c$

We have

$$\lim_{y \to c} [f(y) - f(c)] = \lim_{y \to c} \left[\frac{f(y) - f(c)}{y - c} \times (y-c)\right]$$

$$= \lim_{y \to c} \frac{f(y) - f(c)}{y - c} \lim_{y \to x} (y-c)$$

$$= f'(c) \times 0 = 0$$

so that $\lim_{y \to c} f(y) = f(c)$

$\Rightarrow$ f is continuous at c.

Example:- Example of a function which is continuous at a point but not differentiable at that point.

The function $f : [-1, 1] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = |x|$, is continuous at 0 but not differentiable at 0.

Solution:- Here

$$\lim_{x \to 0+0} f(x) = \lim_{x \to 0-0} f(x) = 0 = f(0)$$

Hence f is continuous at 0.

We have, $$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Thus $$\lim_{x \to 0+0} \frac{f(x) - f(0)}{x - 0} = 1.$$

Hence the right hand derivative $f'(0+0)$ at 0 exists & is equal to 1.

Again $\lim_{x \to 0-0} \frac{f(x) - f(0)}{x - 0} = -1$.

Hence the left hand derivative $f'(0-0)$ at 0 exists & is equal to -1.

Thus $f'(0+0) \neq f'(0-0)$

Hence f is not differentiable at 0 although it is continuous at 0.

Example ④ A function is defined on the real line $\mathbb{R}$ as follows

$$f(x) = x \sin\frac{1}{x}, \quad x \neq 0$$

$$f(0) = 0$$

Show that f is continuous at $x=0$ but not differentiable at $x=0$.

Proof:$\rightarrow$ $\because \left|\sin\frac{1}{x}\right| \leq 1$.

For given $\epsilon > 0$,

$$\left|x \sin\frac{1}{x}\right| = |x|\left|\sin\frac{1}{x}\right| < \epsilon, \text{ provided } |x| < \epsilon := \delta$$

Hence $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$, Also $f(0) = 0$

Thus $\lim_{x \to 0} f(x) = f(0)$.

Therefor, f is continuous at $x=0$.

Now, $\frac{f(x) - f(0)}{x - 0} = \frac{x \sin\frac{1}{x} - 0}{x - 0} = \sin\frac{1}{x}$, for $x \neq 0$.

But $\lim_{x \to 0} \sin\frac{1}{x}$ does not exists.

Hence $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exists.

Therefor f is not differentiable at $x=0$.

Example:→ $f(x) = x^2 \sin\frac{1}{x}$, when $x \neq 0$

$= 0$, when $x = 0$

proved that f is differentiable at $x=0$, while f' is not.

Solution:→ $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x - 0}$

$= \lim_{x \to 0} x \sin\frac{1}{x} = 0$

$\therefore f'(0) = 0$

Hence f is differentiable at $x=0$ & $f'(0)=0$.

For $x \neq 0$.

$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

Now, $\frac{f'(x) - f'(0)}{x - 0} = \frac{2x \sin\frac{1}{x} - \cos\frac{1}{x} - 0}{x - 0}$

$= \left(2 \sin\frac{1}{x} - \frac{1}{x} \cos\frac{1}{x}\right)$

Which does not have a limit as $x \to 0$

$\because \sin\frac{1}{x}$ does not have a limit as $x \to 0$

Hence $f'(x)$ is not differentiable at 0.

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differentiability

Example: → show by an example that the existance of the derivative of a function of a real variable does not guarantee the continuity of the derivative.

solution: →

Let $f(x) = x^2 \cos\frac{1}{x}$, $x \neq 0$

$= 0$, $x = 0$

In this case $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{x \to 0} x \cos\frac{1}{x} = 0$.

Hence $f'(0)$ exists & is equal to zero.

But $f'(x) = 2x \cos\frac{1}{x} + \sin\frac{1}{x}$, for $x \neq 0$

& $\lim\limits_{x \to 0} f'(x)$ does not exists.

Hence f' is not ~~differen~~ continuous at 0 although f' exists at 0.

THANK YOU